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ON THE DIFFUSION OF SPECIES IN SIMILAR BOUNDARY LAYERS WITH CONTINUOUSLY VARYING PROPERTIES

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Abstract—The solutions derived by Freeman and Simpkins for the diffusion equation in an incompressible chemically frozen boundary-layer flow are extended to include variations in surface and fluid properties by a suitable change of variable. In a worked example for the flat plate case, a comparison is made with the exact series solution due to Inger and the local similarity approximation. For large values of the streamwise co-ordinate the local similarity approximation is shown to be identical to the first-order term of the asymptotic expansions derived herein.

NOMENCLATURE

b, c, constants defined in equation (2.13);
D, binary diffusion coefficient;
$f(\eta)$, reduced stream function, see equation (2.4);
$\hat{\imath}$, imaginary unit $= (-1)^{1/2}$;
k, recombination-rate constant;
K, constant defined in equation (3.2);
$l(\eta)$, non-dimensional density–viscosity parameter;
m, power law constant;
Sc, Schmidt number $= (\mu/\rho D)$;
u, streamwise velocity component;
x, y, orthogonal co-ordinate system;
z, reduced mass fraction $= \alpha/\alpha_e$.

Greek symbols

α, dissociation fraction;
$\Gamma(n)$, gamma function;
$\delta(\hat{\imath}s)$, Dirac delta function;
$\zeta(\xi)$, transformed co-ordinate, see equation (2.9);
λ, power law constant;
μ, viscosity;
ξ, transformed co-ordinate, see equation (2.1);
ρ, density;
σ, power law constant.

Subscripts

e, evaluated at the edge of the boundary layer;
0, reference value;
w, conditions on the surface.

INTRODUCTION

THE SOLUTION of the diffusion equation for the class of incompressible Falkner–Skan laminar boundary layers in flows which are chemically frozen has been given recently by Freeman and Simpkins [1]. The problem of the diffusion of species in such a flow reduces to that of solving an equation which is similar to the equation obtained in the thermal boundary layer that has been studied by Lighthill [2]. In that work Lighthill noted that if the Prandtl number was assumed to be large, the thermal boundary layer is much thinner than the velocity boundary layer; so that under these conditions the approximation of Fage and Falkner [3] that the velocity increases linearly with distance from the wall may be used. In the same way, if the assumption is made that the layer in which diffusion occurs is thin compared to the velocity boundary layer then since these thicknesses are proportional to their respective diffusivities this implies that the Schmidt number is large.

Using these approximations of linear velocity profile and large Schmidt number enables the diffusion equation in the Mellin transform plane to be solved in terms of confluent hypergeometric functions.

The purpose of this paper is to show that the above theory can be simply modified to include the cases of continuously varying gas and surface properties. By a suitable change of variable both of these effects can be introduced into the problem in such a way that the equation to be solved is unaltered. The results of this modification are compared in a illustrative example with a solution due to Inger [4] for the flow over a flat plate with continuously varying surface catalycity. The solutions of Inger are an extension of the earlier work by Chambré and Acrivos [5] using a method analogous to the Chapman–Rubesin [6] treatment of the heat-transfer problem to non-isothermal surfaces.

2. FORMULATION OF THE PROBLEM

By defining the independent variable in the usual boundary-layer co-ordinates (ξ, η) where

$$\xi = \int_0^x \rho_e \mu_e u_e \, \mathrm{d}x \tag{2.1}$$

and

$$\eta = \frac{\rho_e \mu_e}{(2\xi)^{1/2}} \int_0^y \left(\frac{\rho_w}{\rho_e}\right) \mathrm{d}y \tag{2.2}$$

we can express the equation for the conservation of species in a chemically frozen two dimensional boundary layer as,

$$lz'' + Sc\, fz' - 2\, Sc\, \xi f' \frac{\partial z}{\partial \xi} = 0. \tag{2.3}$$

Here it is assumed that the Schmidt number Sc is constant, the primes denote differentiation with respect to η, and the following non-dimensional quantities have been defined,

$$f'(\eta) = \frac{u}{u_e(\xi)}; \quad z(\eta) = \frac{\alpha}{\alpha_e}; \quad l(\eta) = \frac{\rho\mu}{\rho_e \mu_e} \tag{2.4}$$

and $l(\eta)$ is assumed to have a constant value of unity. This assumption can be relaxed quite simply since it can be seen from (2.3) that the quotient (l/Sc) can be assumed to be constant, so that $l \equiv 1$ is no longer a necessary condition. Such a procedure might improve the accuracy of the solutions since both Sc and l could vary within the imposed limitation that their quotient remains constant. This could be carried out by introducing $\overline{Sc} = (Sc/l)$ throughout the following derivations without essentially affecting the quoted solutions. Alternatively one can consider l itself as a suitably chosen constant and absorb it into the co-ordinate transformations.

The boundary condition for first order reactions at the wall is given by

$$\left(\frac{\partial z}{\partial \eta}\right)_w = \frac{(2\xi)^{1/2} k_w Sc}{u_e \mu_w} z(\xi, 0). \tag{2.5}$$

We now introduce the variations in properties along the surface in the following manner,

$$\rho_w \mu_w = Bx^\sigma \quad \text{and} \quad \left(\frac{k_w}{\mu_w}\right) = \left(\frac{k_w}{\mu_w}\right)_0 x^\lambda \tag{2.6}$$

and assume the external velocity is given by

$$u_e = Ax^m \tag{2.7}$$

where A, B and $(k_w/\mu_w)_0$ are all constants. In order that equation (2.4) and (2.6) are to remain consistent it is necessary to stipulate the value of σ be compatible with the other assumptions, for example in incompressible flow $\sigma = 0$. Using (2.6) and (2.7), the boundary condition at the wall (2.5) becomes

$$\left(\frac{\partial z}{\partial \eta}\right)_w = \frac{\sqrt{(2)}\, Sc}{A}\left(\frac{k_w}{\mu_w}\right)_0 \left(\frac{1+\sigma+m}{AB}\right)^{\{(\lambda-m)/(1+\sigma+m)\}} \xi^{\{(1+2\lambda+\sigma-m)/[2(1+\sigma+m)]\}}\, z(\xi, 0) \quad (2.8)$$

Replacing the streamwise co-ordinate by a new variable ζ defined as

$$\zeta = \frac{\sqrt{(2)}\, Sc}{A}\left(\frac{k_w}{\mu_w}\right)_0 \left(\frac{1+\sigma+m}{AB}\right)^{\{(\lambda-m)/(1+\sigma+m)\}} \xi^{\{(1+2\lambda+\sigma-m)/[2(1+\sigma+m)]\}} \quad (2.9)$$

the wall boundary condition becomes

$$\frac{\partial z}{\partial \eta}(\zeta, 0) = \zeta z(\zeta, 0) \quad (2.10)$$

and equation (2.3) yields,

$$z'' + f\, Sc\, z' - Sc\, f'\left\{\frac{1+2\lambda+\sigma-m}{1+\sigma+m}\right\}\zeta\frac{\partial z}{\partial \zeta} = 0 \quad (2.11)$$

Making the Fage and Falkner assumption that the velocity at the wall may be approximated by its form in that vicinity, namely as

$$f(\eta) = \tfrac{1}{2}\eta^2 f''(0) \quad (2.12)$$

and introducing the following constants

$$b = \tfrac{1}{6}\, Sc\, f''(0)$$
$$c = \tfrac{3}{2}\left\{\frac{1+m+\sigma}{1+2\lambda+\sigma-m}\right\} \quad (2.13)$$

we can express the diffusion equation (2.11) as

$$z'' + 3b\eta^2 z' - 9\left(\frac{b}{c}\right)\eta\, \zeta \frac{\partial z}{\partial \zeta} = 0 \quad (2.14)$$

where the boundary conditions are given as

$$\left.\begin{aligned} &\frac{\partial z}{\partial \eta}(\zeta, 0) = \zeta\, z(\zeta, 0) \\ \text{and} \quad &z(\zeta, \infty) = 1. \end{aligned}\right\} \quad (2.15)$$

3. GENERAL SOLUTION

The form of the diffusion equation derived in Section 2 as equation (2.14) with the boundary conditions (2.15) is identical to that previously solved by Freeman and Simpkins [1] using Mellin transform techniques. The difference between the two problems is that here the properties of both the gas and the surface may vary continuously with x, whereas previously they had been assumed to be constant. The dependence of these variations on the position along the surface is inherent in (2.14) and (2.15) through the definition of the variable ζ and the constant b.

The complete solution of equations (2.14) and (2.15) has been derived in the earlier paper, and is given in the transform plane as, (see Appendix)

$$\tilde{z}(s,0) = \left\{\left(\frac{2c}{3}\right)^{2/3} K\right\}^s \frac{\Gamma(s)\,\Gamma[\frac{2}{3}(1-s)]}{\Gamma(\frac{2}{3})} \prod_{i=1}^{\infty} \left\{ \frac{\Gamma[(i-s)/c]\,\Gamma[\frac{2}{3}(i+1-s)]\,\Gamma(i/c+\frac{2}{3})\,\Gamma(2i/3)}{\Gamma[(i-s)/c+\frac{2}{3}]\,\Gamma[\frac{2}{3}(i-s)]\,\Gamma(i/c)\,\Gamma[\frac{2}{3}(i+1)]} \right\} \tag{3.1}$$

where

$$K = \frac{[b^{1/3}\,\Gamma(\frac{2}{3})]}{c\,\Gamma(\frac{4}{3})}. \tag{3.2}$$

Equation (3.1) has been constructed such that the pole occurring at $s = 0$ has a residue of unity. The value of the function $z(\zeta, 0)$ in the physical plane may now be obtained from (3.1) by direct application of the Mellin inversion formula

$$z(\zeta, 0) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \tilde{z}(s, 0)\,\zeta^{-s}\,ds. \tag{3.3}$$

Where C lies in the region where $\tilde{z}$ is regular [7]. Now when c is a rational fraction, the infinite product of (3.1) may be simplified considerably as we shall show in the following section.

4. PARTICULAR SOLUTIONS

4.1 *The flat plate* ($m = 0$)

As an example of this technique consider the case of a flat plate with variable surface catalycity, immersed in a frozen dissociated gas stream. For this example then $m = 0$ so that if we choose $\lambda = \frac{1}{2}$ and $\sigma = 0$ we have $c = \frac{3}{4}$ and the solution may be compared with that of Inger.

Substituting $c = \frac{3}{4}$ into (3.1) then yields:

$$\tilde{z}(s,0) = \{(\tfrac{1}{2})^{2/3}K\}^s \frac{\Gamma(s)\,\Gamma[\frac{2}{3}(1-s)]}{\Gamma(\frac{2}{3})} \prod_{i=1}^{p} \left\{ \frac{\Gamma[\frac{4}{3}(i-s)]\,\Gamma[\frac{2}{3}(i+1-s)]\,\Gamma[(4i/3)+\frac{2}{3}]\,\Gamma(2i/3)}{\Gamma[\frac{4}{3}(i-s)+\frac{2}{3})]\,\Gamma[\frac{2}{3}(i-s)]\,\Gamma(4i/3)\,\Gamma[\frac{2}{3}(i+1)]} \right\} \tag{4.1}$$

where we shall allow $p \to \infty$ after simplifying the product. After some manipulation, equation (4.1) can be shown to be

$$\tilde{z}(s,0) = \left\{\frac{16\,K}{9}\right\}^s \frac{\Gamma(\frac{9}{4})\,\Gamma(\frac{3}{2})}{\Gamma(\frac{4}{3})\,\Gamma(\frac{8}{3})} \left\{ \frac{\Gamma(s)\,\Gamma[\frac{4}{3}(1-s)]\,\Gamma[\frac{4}{3}(2-s)]}{\Gamma(\frac{9}{4}-s)\,\Gamma(\frac{3}{2}-s)} \right\}, \tag{4.2}$$

where the behaviour of $\Gamma(ax+b) \sim (ax)^b\,\Gamma(ax)$ for large values of x has been used [8]. In equation (4.2) poles occur along the positive real axis whenever the arguments of one of the two gamma functions in the numerator becomes a negative integer, this occurs when s has the value of either $(1 + \frac{3}{4}n)$ or $(2 + \frac{3}{4}n)$. Thus by substituting these values of s into (4.2) the asymptotic expansion of $z(\zeta, 0)$ can be obtained by evaluating the residues of each of the poles. However, it can be seen from the form of (4.2) that not all of the terms in the expansion for the above values of s will occur, since for some of these values the arguments of the gamma functions in the denominator will themselves become negative integers and hence cancelling will occur. If we now substitute $s = 3s'$ into (4.2) then application of the gamma function duplication formula yields [8],

$$\tilde{z}(3s',0) = \frac{K^{3s'}\,3^{(3s'-13/4)}}{4\pi^2\,4^{(2s'-3)}} \left\{ \frac{\Gamma(\frac{9}{4})\,\Gamma(\frac{3}{2})}{\Gamma(\frac{4}{3})\,\Gamma(\frac{8}{3})} \right\} \frac{\Gamma(s')\,\Gamma(s'+\frac{1}{3})\,\Gamma(s'+\frac{2}{3})\,\Gamma(\frac{1}{3}-s')\,\Gamma(\frac{7}{12}-s')\,\Gamma(\frac{2}{3}-s')\,\Gamma(\frac{11}{12}-s')}{\Gamma(\frac{3}{4}-s')\,\Gamma(\frac{1}{2}-s')} \tag{4.3}$$

The form of $\tilde{z}(3s', 0)$ given in (4.3) shows that poles now occur along the positive real axis whenever s attains any one of the following values, $(n + \frac{1}{3})$, $(n + \frac{7}{12})$, $(n + \frac{2}{3})$ or $(n + \frac{11}{12})$. This expression is, however, more straightforward than (4.2) because now cancelling of terms cannot occur for any of the above values of s since the arguments of the gamma functions in the denominator do not attain negative integer values. Using the identity

$$\Gamma(s')\,\Gamma(s' + \tfrac{1}{3})\,\Gamma(s' + \tfrac{2}{3}) = \frac{4\pi^2}{3}\left\{\frac{\Gamma(s' + \frac{1}{3})}{\Gamma(1 - s')\,\Gamma(\frac{1}{3} - s')} - \frac{\Gamma(s')}{\Gamma(\frac{1}{3} - s')\,\Gamma(\frac{2}{3} - s')} - \frac{\Gamma(s' + \frac{2}{3})}{\Gamma(1 - s)\,\Gamma(\frac{2}{3} - s')}\right\} \tag{4.4}$$

equation (4.3) may be expressed as the sum of three terms each of which has only a single gamma function yielding poles on the negative real axis. Hence

$$\tilde{z}(3s', 0) = \left\{\frac{\Gamma(\frac{9}{4})\,\Gamma(\frac{3}{2})}{\Gamma(\frac{4}{3})\,\Gamma(\frac{8}{3})}\right\}\frac{3^{(3s-17/4)}}{4^{(2s-3)}}\left\{\frac{\Gamma(s' + \frac{1}{3})\,\Gamma(\frac{7}{12} - s')\,\Gamma(\frac{2}{3} - s')\Gamma(\frac{11}{12} - s')}{\Gamma(1 - s')\,\Gamma(\frac{1}{2} - s')\,\Gamma(\frac{3}{4} - s')} - \right.$$
$$\left.\frac{\Gamma(s')\,\Gamma(\frac{7}{12} - s')\,\Gamma(\frac{11}{12} - s')}{\Gamma(\frac{1}{2} - s')\,\Gamma(\frac{3}{4} - s')} - \frac{\Gamma(s' + \frac{2}{3})\,\Gamma(\frac{1}{3} - s')\,\Gamma(\frac{7}{12} - s')\,\Gamma(\frac{11}{12} - s')}{\Gamma(1 - s')\,\Gamma(\frac{1}{2} - s')\,\Gamma(\frac{3}{4} - s')}\right\} \tag{4.5}$$

Taking the inverse transform then allows (4.5) to be represented as a sum of three generalized hypergeometric functions [8]. Thus in the real plane we obtain,

$$z(\zeta^3, 0) = \frac{\Gamma(\frac{9}{4})\,\Gamma(\frac{3}{2})}{\Gamma(\frac{4}{3})\,\Gamma(\frac{8}{3})}\left\{\frac{\Gamma(\frac{11}{12})\,\Gamma(\frac{15}{12})}{\Gamma(\frac{4}{3})\,\Gamma(\frac{5}{6})\,\Gamma(\frac{13}{12})}\,\frac{4^{11/3}}{3^{21/4}}\left(\frac{\zeta}{K}\right){}_3F_3\left(\begin{matrix}\frac{11}{12}, 1, \frac{15}{12}\\ \frac{4}{3}, \frac{5}{6}, \frac{13}{12}\end{matrix}; \Omega\right) - \right.$$
$$\left.\frac{\Gamma(\frac{7}{12})\,\Gamma(\frac{11}{12})4^3}{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{4})\,3^{17/4}}\,{}_2F_2\left(\begin{matrix}\frac{7}{12}, \frac{11}{12}\\ \frac{1}{2}, \frac{3}{2}\end{matrix}; \Omega\right) - \frac{\Gamma(\frac{15}{12})\,\Gamma(\frac{19}{12})}{\Gamma(\frac{5}{3})\,\Gamma(\frac{7}{6})\,\Gamma(\frac{17}{12})}\,\frac{4^{13/4}}{3^{25/4}}\left(\frac{\zeta}{K}\right)^2{}_3F_3\left(\begin{matrix}1, \frac{15}{12}, \frac{19}{12}\\ \frac{5}{3}, \frac{7}{6}, \frac{17}{12}\end{matrix}; \Omega\right)\right\} \tag{4.6}$$

where

$$\Omega = -\frac{16}{27}\left(\frac{\zeta}{K}\right)^3$$

Equation (4.6) then represents a complete series solution for all ζ. Alternatively one can find the expansion about $\zeta = 0$ and the asymptotic expansion by deforming the contour in the transform plane to embrace all the poles of $\tilde{z}(3s', 0)$ [1]. Using this technique the expansion about $\zeta = 0$ may be obtained by putting

$$s = -n, \quad s = -(n + \tfrac{1}{3}) \quad \text{and} \quad s = -(n + \tfrac{2}{3})$$

into (4.3) and then evaluating the residues of the poles of the transform that occur for the above values of s. This then yields the expansion about $\zeta = 0$ as,

$$z(\zeta, 0) = \left(\frac{1}{2\pi}\right)^2\frac{\Gamma(\frac{9}{4})\,\Gamma(\frac{3}{2})}{\Gamma(\frac{4}{3})\,\Gamma(\frac{8}{3})}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\{\frac{4^{(2n+3)}}{3^{(3n+13/4)}}\left[\frac{\Gamma(\frac{1}{3} - n)\,\Gamma(\frac{2}{3} - n)\,\Gamma(\frac{1}{3} + n)\,\Gamma(\frac{7}{12} + n)\,\Gamma(\frac{2}{3} + n)}{\Gamma(\frac{3}{4} + n)\Gamma(\frac{1}{2} + n)}\right]\times\right.$$
$$\Gamma(\tfrac{11}{12} + n)\left(\frac{\zeta}{K}\right)^{3n} + \frac{4^{(2n+11/3)}}{3^{(3n+17/4)}}\left[\frac{\Gamma(-n - \frac{1}{3})\,\Gamma(\frac{1}{3} - n)\,\Gamma(n)\,\Gamma(n + \frac{11}{12})\,\Gamma(n + 1)\,\Gamma(n + \frac{5}{4})}{\Gamma(\frac{13}{12} + n)\,\Gamma(\frac{5}{6} + n)}\right]\left(\frac{\zeta}{K}\right)^{3n+1} +$$
$$\left.\frac{4^{(2n+13/3)}}{3^{(3n+21/4)}}\left[\frac{\Gamma(-n - \frac{2}{3})\,\Gamma(-n - \frac{1}{3})\,\Gamma(n + 1)\,\Gamma(n + \frac{5}{4})\,\Gamma(n + \frac{4}{3})\,\Gamma(n + \frac{19}{12})}{\Gamma(n + \frac{17}{12})\,\Gamma(n + \frac{7}{6})}\right]\left(\frac{\zeta}{K}\right)^{3n+2}\right\} \tag{4.7}$$

The asymptotic expansion is obtained by evaluating the residues of the transform when s has the following values,

$$s = (n + \tfrac{1}{3}), \quad (n + \tfrac{7}{12}), \quad (n + \tfrac{2}{3}) \quad \text{and} \quad (n + \tfrac{11}{12})$$

This yields the asymptotic form as:

$$\begin{aligned} z(\zeta, 0) = {} & \frac{1}{4\pi^2}\left[\frac{\Gamma(\frac{9}{4})\,\Gamma(\frac{3}{2})}{\Gamma(\frac{4}{3})\,\Gamma(\frac{8}{3})}\right] \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\{ \frac{3^{(3n-13/4)}}{4^{(2n-7/3)}} \left[\frac{\Gamma(n+\frac{1}{3})\,\Gamma(n+\frac{2}{3})\,\Gamma(n+1)}{\Gamma(\frac{5}{12}-n)\,\Gamma(\frac{1}{6}-n)}\right] \times \right. \\ & [\Gamma(\tfrac{1}{3}-n)\,\Gamma(\tfrac{7}{12}-n)\,\Gamma(\tfrac{1}{4}-n)] \left(\frac{K}{\zeta}\right)^{(3n+1)} + \frac{3^{(3n-3/2)}}{4^{(2n-11/6)}} \left[\frac{\Gamma(n+\frac{7}{12})\,\Gamma(n+\frac{11}{12})\,\Gamma(n+\frac{5}{4})}{\Gamma(\frac{1}{6}-n)\,\Gamma(-\frac{1}{12}-n)}\right] \times \\ & [\Gamma(-\tfrac{1}{4}-n)\Gamma(\tfrac{1}{12}-n)\,\Gamma(\tfrac{1}{3}-n)] \left(\frac{K}{\zeta}\right)^{(3n+7/4)} + \frac{3^{(3n-11/4)}}{4^{(2n-5/3)}} \left[\frac{\Gamma(n+\frac{2}{3})\Gamma(n+1)\,\Gamma(n+\frac{4}{3})\,\Gamma(-\frac{1}{3}-n)}{\Gamma(\frac{1}{12}-n)\,\Gamma(-\frac{1}{6}-n)}\right] \times \\ & [\Gamma(-\tfrac{1}{12}-n)\,\Gamma(\tfrac{1}{4}-n)] \left(\frac{K}{\zeta}\right)^{(3n+2)} + \frac{3^{(3n-1/2)}}{4^{(2n-7/6)}} \left[\frac{\Gamma(n+\frac{11}{12})\,\Gamma(n+\frac{5}{4})\,\Gamma(n+\frac{19}{12})\,\Gamma(-\frac{7}{12}-n)}{\Gamma(-\frac{1}{6}-n)\,\Gamma(-\frac{5}{12}-n)}\right] \times \\ & \left. [\Gamma(-\tfrac{1}{3}-n)\,\Gamma(-\tfrac{1}{4}-n)] \left(\frac{K}{\zeta}\right)^{(3n+11/4)} \right\} \end{aligned} \tag{4.8}$$

It can easily be seen that the expansion about $\zeta = 0$ could have been obtained in the same manner from (4.2) or indeed by direct substitution into the solutions of the earlier paper. In the case of the asymptotic expansion we have seen from (4.2) that poles occur along the positive real axis whenever $s = (i + nc)$ for values of $i = 1$ and 2. When n is non-zero the values $s_n = nc$ are eigenvalues, and the corresponding solutions will be the eigen solutions of equation (2.14). If a direct substitution procedure is adopted for the asymptotic expansion the appropriate form is

$$z = \sum_{n=0} \sum_{k=0} \frac{L_{kn}(\eta)}{\zeta^{k+nc}}.$$

When $k = 0$ and $n \neq 0$ the boundary conditions require that $L_{on}(\infty) = L_{on}(0) = 0$. The functions $L_{on}(\eta)$ are then the eigen solutions mentioned above. The form of expansion becomes

$$z = \sum_{k=0} \frac{L_{ko}(\eta)}{\zeta^k} + \sum_{k=1} \sum_{n=1} \frac{A_n\, L'_{kn}(\eta)}{\zeta^{k+sn}},$$

when z is made to satisfy the boundary conditions where the $L'_{kn}(\eta)$ are known explicitly. Such procedure does not yield explicit values for the constants A_n, and it is necessary to use methods similar to those outlined previously to evaluate the terms in each particular case.

The series expansion about $\zeta = 0$ has been evaluated for the first fifteen terms and the results are shown in Fig. 1 compared with Inger's fifteen term solution for this particular case. The asymptotic expansion result is also shown in Fig. 1, here the terms have been evaluated up to $0(K/\zeta)^5$.

4.2 *Stagnation point flow* ($m = 1$)

Two interesting results arise in the case of the stagnation point flow for which c is in general given by $c = \frac{3}{2}[(2 + \sigma)/(2\lambda + \sigma)]$. The first case is that in which $\sigma = 0$ and $\lambda = 2$ so that $c = \frac{3}{4}$ and the solutions given in the previous section may be used directly, bearing in mind that the behaviour of the surface catalycity is now proportional to x^2 as opposed to $x^{1/2}$.

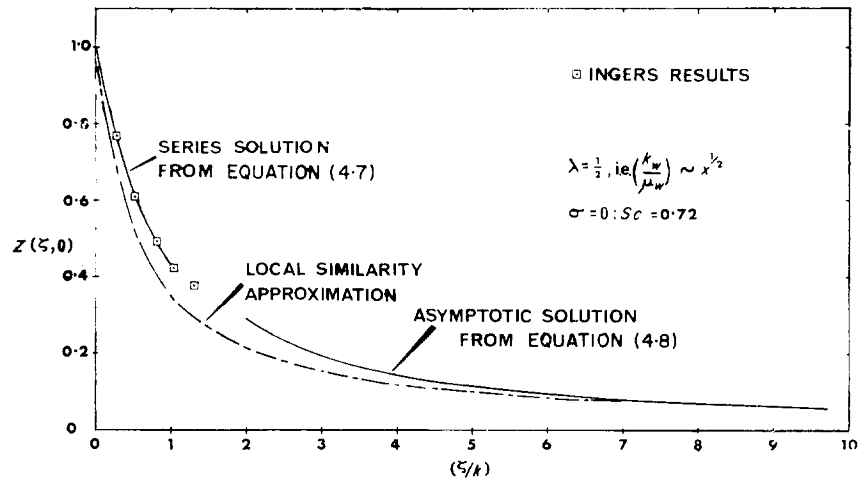

FIG. 1. Comparison of the atom concentration along a plate for a power law distribution of surface catalycity with Inger's series solution.

The second case is that in which the atomic recombination reaction at the wall varies linearly with distance from the stagnation point (i.e. $\lambda = 1$).* Under these conditions the value of $c = \frac{3}{2}$ and is independent of the parameter σ, so that the incompressible solutions may be applied to the compressible flows. With $c = \frac{3}{2}$ we find that the required solution is identical to the flat plate solution with $\lambda = 0$, which has been outlined in the earlier paper. Thus in this particular case, the expansion about $\zeta = 0$ is given by

$$z(\zeta, 0) = \sum_{n=0} \frac{(-1)^n}{n!} \left(\frac{\zeta}{K}\right)^n \frac{\Gamma[\frac{2}{3}(n+1)]}{\Gamma(\frac{2}{3})}$$

and the asymptotic behaviour as

$$z(\zeta, 0) = \tfrac{3}{2} \sum_{n=0} \frac{(-1)^n}{n!} \left(\frac{K}{\zeta}\right)^{1+(3n/2)} \left\{\frac{\Gamma[1+(3n/2)]}{\Gamma(\frac{2}{3})}\right\}$$

Clearly, although this solution is identical to the flat plate solution with $\lambda = 0$, it will be scaled accordingly due to the difference of the values of the variable ζ in the two cases.

5. THE LOCAL SIMILARITY APPROXIMATION

The concept of local similarity is often invoked in order that a set of similar solutions may be extended to regions where similarity no longer exists. This approximation has been discussed in general by Hayes and Probstein [9] and its application to the present problem has been considered by Inger [4], so that only the relevant details will be given here.

When the non-similar term is either small enough to be neglected or does not exist in equation (2.3), the equation can be formally integrated to yield the result first given by Goulard [10], that

$$z(\zeta, \eta) = \frac{1 + \zeta_o I_1(\eta)}{1 + \zeta_o I_1(\infty)} \tag{5.1}$$

* Clearly, in this case since λ is odd, one is strictly speaking not obtaining a solution to a symmetric two dimensional stagnation point since then all surface properties like k_w should be even functions of x, i.e. $k_w(x) = k_w(-x)$. The author is grateful to one of the referees for raising this point.

The boundary condition at the wall requires that

$$z'(\zeta, 0) = \zeta_o z(\zeta, 0) = \frac{\zeta_o}{1 + \zeta_o I_1(\infty)} \tag{5.2}$$

where $\zeta = \zeta_o = \text{const.}$, and

$$I_1(\eta) = \int_0^\eta \{\exp[-Sc \int_0^\eta f \, d\eta]\} \, d\eta \tag{5.3}$$

Inger has shown how to ascertain under what conditions the flow properties may be assumed to be slowly varying, by considering the integral formulation of equation (2.3) including the non-similar term. The resulting expression after the locally similar value for $z(\eta)$ given by equation (5.1) has been substituted into the quadrature, is given by the following expression,

$$z(0, \xi) \doteq \frac{1 + \dfrac{2\, Sc\, \xi\, (\partial\zeta/\partial\xi)\, I_1(\infty)}{[1 + I_1(\infty)\, \zeta]^2} [I_3(\infty) - I_2(\infty)]}{[1 + \zeta\, I_1(\infty)]} \tag{5.4}$$

where

$$I_2(\eta) = \int_0^\eta \exp[-Sc \int_0^{\eta_4} f(\eta_3) \, d\eta_3] \left\{ \int_0^{\eta_4} [\exp(Sc \int_0^{\eta_2} f(\eta_1) \, d\eta_1] \times \frac{I_1(\eta_2)}{I_1(\infty)} f'(\eta_2) \, d\eta_2 \right\} d\eta_4$$

and

$$I_3(\eta) = \int_0^\eta \exp[-Sc \int_0^{\eta_4} f(\eta_3) \, d\eta_3] \left\{ \int_0^{\eta_4} [\exp(Sc \int_0^{\eta_2} f(\eta_1) \, d\eta_1] f'(\eta_2) \, d\eta_2 \right\} d\eta_4$$

From equation (5.4) it can be seen that the local similarity approximation will be valid when the second term in the numerator is small compared with unity, that is when $\zeta \gg 1$; such a condition implies that the surface is locally highly catalytic. If, however, ζ is of order unity the above condition is not necessarily true, so that the accuracy of the local similarity approximation becomes more suspect as the surface recombination rate decreases.

Let us now turn our attention to examining the effect of assuming $\zeta \gg 1$ on the local similarity approximation. From equation (5.1) we find that the asymptotic behaviour of $z(0, \zeta)$ for large ζ is given by

$$z(0, \zeta) \doteq \frac{1}{\zeta\, I_1(\infty)} \tag{5.5}$$

Now applying the Fage and Falkner assumption to equation (5.3) we find that

$$I_1(\infty) = \int_0^\infty \{\exp[-\tfrac{1}{6}\, Sc\, \eta^3 f''(0)]\} \, d\eta \tag{5.6}$$

Changing the variable of equation (5.6) it is a straightforward procedure to show that

$$I_1(\infty) = b^{-1/3}\, \Gamma(\tfrac{4}{3}) \tag{5.7}$$

so that the asymptotic behaviour of the local similarity approximation is given by

$$z(0, \zeta) \simeq \frac{b^{1/3}}{\Gamma(\frac{4}{3})\, \zeta} \tag{5.8}$$

where b is defined by equation (2.13).

Now the asymptotic series can be interpreted from equation (4.2) in the following form,

$$z(\zeta, 0) = \frac{9}{16} \frac{\Gamma(\frac{5}{4})\,\Gamma(\frac{2}{3})}{\Gamma(\frac{5}{3})\,\Gamma(\frac{4}{3})} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\{ \frac{\Gamma(1 + 3n/4)\,\Gamma[\frac{4}{3}(1 - 3n/4)]}{\Gamma[(5 - 3n)/4]\,\Gamma[\frac{1}{2}(1 - 3n/2)]} \left(\frac{16}{9}\frac{K}{\zeta}\right)^{1+(3n/4)} + \frac{\Gamma(2 + 3n/4)\,\Gamma[-\frac{4}{3}(1 + 3n/4)]}{\Gamma[(1 - 3n)/4]\,\Gamma[-\frac{1}{2}(1 + 3n/2)]} \left(\frac{16}{9}\frac{K}{\zeta}\right)^{2+(3n/4)} \right\} \tag{5.9}$$

Evaluating the first-order term in this expansion we obtain

$$z(\zeta, 0) \simeq \frac{1}{2\,\Gamma(\frac{5}{3})} \left(\frac{K}{\zeta}\right) + \dots$$

and hence substituting for K from the definition in equation (3.2) we find the first-order term in the asymptotic expansion is given by

$$z(\zeta, 0) \simeq \frac{b^{1/3}}{\Gamma(\frac{4}{3})\,\zeta} + \dots$$

Thus the behaviour obtained from the local similarity approximation for $\zeta \gg 1$ is the same as the first-order term in the asymptotic expansion. The results obtained from using the local similarity approximation have been plotted on Fig. 1. As is to be expected from the above analysis this approximation yields a rather poor quantitative agreement of the atom concentration profile for values of $(\zeta/k) < 5$. After this value the asymptotic behaviour becomes more and more dominated by the first-order term and the agreement predicted above for $\zeta \gg 1$ can be clearly seen. It is also quite clear that because of the above reasons the local similarity solution does not represent a means of effectively bridging the small ζ and large ζ solutions, the error in this vicinity being approximately 25 per cent of the local atom concentration for the case under consideration.

6. CONCLUDING REMARKS

It has been shown that the solutions derived by Freeman and Simpkins for the diffusion equation in an incompressible laminar boundary layer that is chemically frozen may be extended to include continuous variations in both surface and fluid properties adjacent to the wall by a suitable change of variable. The application of the transform technique allows the asymptotic expansion to be evaluated explicitly including the terms involving the eigen values. When the problem is formulated as an integral equation the initial terms in the asymptotic expansion may still be evaluated, viz. Lighthill [2]; however, the evaluation of the terms which include the eigen solutions is more difficult.

The results of this method have been compared in a particular case with Inger's series solution of the exact equation and the local similarity approximation. It has been shown that the results obtained here appear to coincide with those of Inger for values of (ζ/K) less than unity, while the local similarity approximation predicts atom concentration values which are some 25 per cent too low in this region. For values of $\zeta \gg 1$ the local similarity approximation gives the first-order term in the asymptotic expansion. The good agreement between the present solution and Ingers exact results is not altogether surprising since the approximation of linear velocity profile near the wall is valid for Schmidt number of order unity, and is, of course, asymptotically exact in the limit $Sc \to \infty$. It is well known that the Schmidt number is a function of the atom concentration, and that for $\alpha_e \ll 1$ the value of $Sc \simeq 0{\cdot}5$, see Dorrance [11], so that the assumption of Sc large might appear to be questionable. Recent work by Freeman [12] suggests that even for values of $Sc \simeq 0{\cdot}5$ the first eigen value in the asymptotic solution has only increased by approximately 5 per cent of the value for $Sc \gg 1$. However, when $Sc < 0{\cdot}5$ the eigen value increases steeply the limiting value of 2 when $Sc \equiv 0$. This behaviour is due to the non-uniformity present in the solution as $Sc \to 0$. Thus the approximation of Sc large appears to be good for values of $Sc \geqslant 0{\cdot}5$.

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APPENDIX

Derivation of the solution to equation (2.14)

In the present notation the formal definition of a Mellin transform is given by Erdélyi as [7]

$$\tilde{z}(s, \eta) = \int_0^\infty z(\zeta, \eta)\, \zeta^{s-1}\, d\zeta \tag{A.1}$$

Applying this definition to equation (2.14) and using the transform property that

$$[-\zeta\,(\partial z/\partial\zeta)] \doteqdot s\tilde{z},$$

we obtain the equation in the Mellin plane as

$$\tilde{z}'' + 3b\eta^2\,\tilde{z}' + 9\,(b/c)\,\eta s\,\tilde{z} = 0 \tag{A.2}$$

The complete solution of an equation of this type has been given by Punnis [13] and Fettis [14] in terms of the confluent hypergeometric function. Formally this may be achieved by the substitution of $\tilde{z}(s) = e^{-\theta}\,Y(\theta)$ into equation (A.2), where $\theta = b\eta^3$. The resulting equation is then recognisable as one of the confluent hypergeometric type, the solution of which is,

$$\tilde{z}(s, \eta) = A \exp[-b\eta^3]\,\{{}_1F_1\,[\tfrac{2}{3} - (s/c);\, \tfrac{2}{3};\, b\eta^3] + (B/A)\,b^{1/3}\eta\;{}_1F_1\,[1 - (s/c);\, \tfrac{4}{3};\, b\eta^3]\} \tag{A.3}$$

see for example, Erdélyi [8]. The boundary conditions at $\eta = 0$ and $\eta \to \infty$ then yield the values of A and B so that the solution can be expressed as

$$\tilde{z}(s, \eta) = 2\pi\delta(\hat{\imath}s) - \frac{\Gamma(\frac{4}{3})\,\Gamma[\frac{2}{3} - (s/c)]}{b^{1/3}\,\Gamma(\frac{2}{3})\,\Gamma[1 - (s/c)]}\left(\frac{\partial \tilde{z}}{\partial \eta}\right)_{n=0} \exp[-b\eta^3] \times$$
$$\left\{ {}_1F_1\,[\tfrac{2}{3} - (s/c); \tfrac{2}{3}; b\eta^3] - \frac{\Gamma(\frac{2}{3})\,\Gamma[1 - (s/c)]}{\Gamma(\frac{4}{3})\,\Gamma[\frac{2}{3} - (s/c)]} b^{1/3}\eta\, {}_1F_1\,[1 - (s/c); \tfrac{4}{3}; b\eta^3] \right\} \quad \text{(A.4)}$$

The boundary condition that $z \to 1$ as $\eta \to \infty$ is satisfied since $\tilde{z} \to 2\pi\delta(\hat{\imath}s)$. The application of the transform formula that $\zeta z(\zeta) \doteqdot \tilde{z}(s + 1)$ yields the boundary condition at $\eta = 0$ as

$$\tilde{z}(s + 1, 0) = \frac{\partial \tilde{z}}{\partial \eta}(s, 0) \quad \text{(A.5)}$$

When this condition is imposed on equation (A.4) we obtain the behaviour of $\tilde{z}$ on the surface in the following recurrence relationship,

$$\tilde{z}(s, 0) = 2\pi\,\delta(\hat{\imath}s) - \frac{\Gamma(\frac{4}{3})\,\Gamma[\frac{2}{3} - (s/c)]}{b^{1/3}\,\Gamma(\frac{2}{3})\,\Gamma[1 - (s/c)]}\tilde{z}(s + 1, 0) \quad \text{(A.6)}$$

Rearranging (A.6) and neglecting for the present the delta function which only has a contribution at $s = 0$, we can express the recurrence equation as

$$\tilde{z}(s, 0) = \frac{K(s - 1)\,\Gamma[(1 - s)/c]\,\tilde{z}(s - 1)}{\Gamma[\frac{2}{3} + (1 - s)/c]} \quad \text{(A.7)}$$

In equation (A.7) it can be seen that a pole will occur when the gamma function in the numerator has a value $s = (1 + nc)$ for integer n, and therefore by the recurrence equation a series of poles will occur along the positive real axis at values of $s = (i + nc)$ where $1 \leqslant i \leqslant \infty$. Now, the change in the boundary condition at the leading edge of the plate or wedge requires the stipulation that $s\tilde{z}\,(s, 0) \to 1$ as $s \to 0$, this condition is met by the introduction of the delta function. Since a pole therefore occurs at the origin, a series of poles will be generated along the real axis for both positive and negative integer values of s as a result of equation (A.6).

If equation (A.7) is examined for various values of s it is found that a formal solution of the equation can be expressed as

$$\tilde{z}(s, 0) = K^s\,\Gamma(s) \prod_{i=1}^{\Omega} \left\{ \frac{\Gamma[(i - s)/c]\,\Gamma[(i/c) + \frac{2}{3}]}{\Gamma[(i - s)/c + \frac{2}{3}]\,\Gamma(i/c)} \right\} \quad \text{(A.8)}$$

The product occurring in (A.8) is not absolutely convergent when Ω tends to infinity, and it is necessary to introduce extra terms to satisfy this condition; the resulting expression after this procedure has been carried out corresponds to equation (3.1) in the text.

Résumé—Les solutions obtenues par Freeman et Simpkins pour l'équation de la diffusion dans une couche limite incompressible et figée chimiquement sont étendues pour inclure les variations de propriétés de la surface et du fluide par un changement convenable de variable. Dans un exemple résolu pour le cas de la plaque plane, on fait une comparison avec la solution exacte par série due à Inger et l'approximation de la similitude locale. Pour de grandes valeurs de la coordonnée prise le long de l'écoulement on montre que l'approximation de la similitude locale est identique au terme du premier ordre des développements asymptotiques obtenus ici.

Zusammenfassung—Die von Freeman und Simpkins abgeleiteten Lösungen der Gleichung über die Diffusion in einer chemisch eingefrorenen Grenzschichtströmung bei inkompressiblen Medien werden erweitert, um Veränderungen an der Oberfläche und den Stoffwerten der Flüssigkeit durch einen geeigneten Parameterwechsel mit zu erfassen. An einem für den Fall der ebenen Platte ausgearbeiteten

Beispiel wird die genaue Lösungsreihe von Inger mit der örtlichen Näherung nach der Ähnlichkeitstheorie verglichen. Für grosse Werte der Strömungskoordinate wird gezeigt, dass die örtliche Näherung nach der Ähnlichkeitstheorie identisch ist mit dem Glied 1. Ordnung nach der hier abgeleiteten asymptotischen Reihe.

Аннотация—Решения уравнения диффузии в пограничном слое несжимаемого химически «замороженного» течения, полученные ранее Фрименом и Симпкинсом, распространены на случай непостоянных свойств поверхности и жидкости путем соответствующей модификации переменной. В разбираемом примере плоской пластины дается сравнение с точным решением Ингера в виде ряда и приближением локального подобия. Показано, что для больших значений продольной координаты приближение локального подобия тождественно члену первого порядка асимптотических разложений, полученных в этой статье.